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Quantum Thermal Effect of Nonstatic Charged Black Hole

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The difficulty of calculating energy-momentum tensors is avoided by finding directly the solution of Klein–Gordon and Dirac equations near the horizon. Both the location of the event horizon and the Hawking radiation temperature of a nonstatic charged black hole are shown. The results indicate that the Hawking radiation temperature can be regarded as a compensating effect under the time-scale transformation.

Ever since Hawking's discovery of the quantum radiation of the black hole, much work has been done concerning the backreaction of the quantum radiation on the black hole. Almost all the work carried out in this area first finds a renormalized energy-momentum tensor as the source term. By means of this method, we can only obtain the approximate value of the Hawking radiation temperature for spherically symmetric black holes after considering the backreaction [1, 2]. In this article, the difficulty of calculating the energy-momentum tensor is avoided by directly finding the solution of Klein–Gordon and Dirac equations near the horizon, with v_* as time coordinate. By means of the generalized tortoise coordinates the KG and Dirac equations are reduced to the standard wave equation. Thus the equation for determining the location of the event horizon can be automaticaly obtained. Following Damour and Ruffini [3] and Sannan [4], by studying the wave function, both the radiation spectrum and the Hawking radiation temperature are obtained, and the physical mechanism for producing the Hawking radiation temperature is distinctly shown. We choose units $\hbar = C = G = K_B = 1$ [5].

The metric of the spherically symmetic charged evaporating black hole in Vaidya–Bonner space–time is as follows [6]:

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$$ds^2 = -B(r, v)\, dv^2 + 2dv\, dr + r^2(d\theta^2 + \sin^2 d\varphi^2) \tag{1}$$

where $B(r, v) = 1 - 2m(v)/r + Q^2(v)/r^2$.

The KG equation which describes the scalar quantity field is

$$\frac{1}{\sqrt{-g}}\left[\left(\frac{\partial}{\partial x^\mu} - ieA_\mu\right)\sqrt{-g}g^{\mu\nu}\left(\frac{\partial}{\partial x^\nu} - ieA_\nu\right)\right]\Phi - \mu^2\Phi = 0 \tag{2}$$

where $A_\mu = (-Q/r, 0, 0, 0)$, and μ is the mass of the KG particle.

Let $\Phi = R(r, v)Y(\theta, \varphi)$. Equation (2) can be reduced to

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0 \tag{3}$$

$$r^2\frac{\partial^2 R}{\partial v\partial r} + \frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial v}\right) + \frac{\partial}{\partial r}\left(r^2 A\frac{\partial R}{\partial r}\right)$$

$$-ieA_0 r^2\frac{\partial R}{\partial r} - ie\frac{\partial}{\partial r}(r^2 A_0 R) - (\mu^2 r^2 + \lambda)R = 0 \tag{4}$$

$$\lambda = l(l+1), \qquad l = 0, 1, 2, 3, \ldots$$

Assuming $R = r^{-1}\rho(r, v)$, we can reduce Eq. (4) to

$$B\frac{\partial^2\rho}{\partial r^2} + 2\frac{\partial^2\rho}{\partial r\partial v} + 2\left(\frac{m}{r^2} - \frac{Q^2}{r^3} + \frac{ieQ}{r}\right)\frac{\partial\rho}{\partial r}$$

$$-\left[\frac{2}{r^2}\left(\frac{m}{r} - \frac{Q^2}{r^{\bar{r}}}\right) + \mu^2 + \frac{l(l+1)}{r^2} + \frac{ieQ}{r^2}\right]\rho = 0 \tag{5}$$

By means of the coordinate transformation [7]

$$r_* = ln[r - r_+(v)], \qquad v_* = \int [B'(r_+, v) + \omega_0/\omega]dv \tag{6}$$

we can be write (5) as

$$\frac{B - 2\dot{r}_+}{r - r_+}\frac{\partial^2\rho}{\partial r_*^2} + 2\left[B'(r_+, v) + \frac{\omega_0}{\omega}\right]\frac{\partial^2\rho}{\partial r_*\partial v_*}$$

$$+\left[\frac{2\dot{r}_+ - B}{r - r_+} + 2\left(\frac{m}{r^2} - \frac{Q^2}{r^3} + \frac{ieQ}{r}\right)\right]\frac{\partial\rho}{\partial r_*}$$

$$-(r - r_+)\left[\frac{2}{r^2}\left(\frac{m}{r} - \frac{Q^2}{r^2}\right) + \mu^2 + \frac{l(l+1)}{r^2} + \frac{ieQ}{r^2}\right]\rho = 0 \tag{7}$$

where r_+ is the location of the event horizon outside the black hole, $B'(r_+, v) = \partial B/\partial r = 2m(v)/r_+^2 - 2Q^2(v)/r_+^3$, $\omega_0 = eV$, and $V = Q/r_+$ is the potential on the surface of the horizon; ω is a constant.

Now, we study Eq. (7) near the event horizon. First we study the coefficient of $\partial^2\rho/\partial r_*^2$;

$$\frac{B - 2\dot{r}_+}{r - r_+} \tag{8}$$

Making use of the null-surface condition [8]

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0 \tag{9}$$

$$f = f(r, v) = 0 \tag{10}$$

we know that

$$[B - 2\dot{r}_+]_{r=r_+} = 0 \tag{11}$$

Therefore

$$\lim_{r\to r_+}\frac{B - 2\dot{r}_+}{r - r_+} = B'(r_+, v) \tag{12}$$

Equation (7) can be reduced to

$$B'(r_+, v)\frac{\partial^2\rho}{\partial r_*^2} + 2\left[B'(r_+, v) + \frac{\omega_0}{\omega}\right]\frac{\partial^2\rho}{\partial r_*\,\partial v_*} + 2i\omega_0\frac{\partial\rho}{\partial r_*} = 0 \tag{13}$$

The solutions of Eq. (13) are

$$\rho_{\text{in}} = e^{-i\omega v_*} \tag{14}$$

$$\rho_{\text{out}} = e^{-i\omega v_*}\, e^{2i\omega r_*} = e^{-i\omega v_*}\,(r - r_+)^{2i\omega} \tag{15}$$

Clearly, ρ_{out} is not analytic at $r = r_+$, but we can extend it by analytic continuation to the inside of the horizon through the lower half complex r plane as in ref 3.

Hence

$$\rho_{\text{out}} \to \tilde{\rho}_{\text{out}} = \rho_{\text{out}}\, e^{2\pi\omega} \tag{16}$$

Following Sannan [4], the scattering probability of the outgoing wave at the horizon is

$$\left|\frac{\rho_{\text{out}}}{\tilde{\rho}_{\text{out}}}\right|^2 = e^{-4\pi\omega} \tag{17}$$

The radiation spectrum and the radiation temperature are, respectively,

$$N_\omega = \frac{1}{e^{\omega/T_{v_*}} - 1} \tag{18}$$

$$T_{v_*} = \frac{1}{4\pi} \tag{19}$$

From (15) we see that ω in (19) is the radiation frequency of the black body when the time coordinate is v_*. Let $\tilde{\omega}$ be the radiation frequency when the time coordinate is v. Then

$$\omega v_* = \tilde{\omega} v \tag{20}$$

We have

$$\omega = \frac{\tilde{\omega} - \omega'}{\kappa}, \qquad \omega' = \frac{1}{v}\int \omega_0 dv, \qquad \kappa = \frac{1}{v}\int B'\,(r_+, v)\,dv \tag{21}$$

Hence, when the time coordinate is v, both the Hawking radiation spectrum and the radiation temperature of a nonstatic charged black hole are

$$N_{\tilde{\omega}} = \frac{1}{e^{(\omega-\omega')/T_v} - 1} \tag{22}$$

$$T_v = \frac{1}{4\pi v}\int B'(r_+, v)\,dv = \frac{1}{2\pi v}\int \left(\frac{m(v)}{r_+^2} - \frac{Q^2(v)}{r_+^3}\right) dv \tag{23}$$

where r_+ is given by (11). Its approximate solution is

$$r_+ \approx \frac{m + \sqrt{m^2 - Q^2(1 - 2\dot{r}_A)}}{1 - 2\dot{r}_A} \tag{24}$$

$$\dot{r}_A = \dot{m} + \frac{m\dot{m} - Q\dot{Q}}{\sqrt{m^2 - Q^2}} \tag{25}$$

The dynamic behavior of the spinor particle (mass is μ_0) in this space–time is described by the Dirac equation [9]

$$\sqrt{2}(\nabla_{ab} + ieA_{ab})P^a + i\mu_0\overline{Q}_b = 0 \tag{26}$$

$$\sqrt{2}(\nabla_{ab} - ieA_{ab})Q^a + i\mu_0\overline{P}_b = 0 \tag{27}$$

where P^a, Q^a, and ∇_{ab} are, respectively, the 2-component and the covariant spinor differentiation expressed with spinor base components.

Let l_μ, n_μ, m_μ, and $\overline{m}_\mu$ be the null tetrad vectors

$$l^\mu l_\mu = n^\mu n_\mu = m^\mu n_\mu = \overline{m}^\mu \overline{m}_\mu = 0, \qquad l^\mu n_\mu = -m^\mu \overline{m}_\mu = 1$$
$$l^\mu m_\mu = l^\mu \overline{m}_\mu = n^\mu m_\mu = n^\mu \overline{m}_\mu = 0 \tag{28}$$
$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu$$

with the signature 2. The spinor forms of Eqs. (26) and (27) are

$$(D + \varepsilon - \rho + ie\mathbf{A} \cdot \mathbf{l})F_1 + (\overline{\delta} + \pi - \alpha + ie\mathbf{A} \cdot \overline{\mathbf{m}})\, F_2 = i \frac{\mu_0}{\sqrt{2}}\, G_1$$

$$(\nabla + \mu - \gamma + ie\mathbf{A} \cdot \mathbf{n})F_2 + (\delta + \beta - \tau + ie\mathbf{A} \cdot \mathbf{m})G_2 = i \frac{\mu_0}{\sqrt{2}}\, G_2 \tag{29}$$

$$(D + \overline{\varepsilon} - \overline{\rho} + ie\mathbf{A} \cdot \mathbf{l})G_2 - (\delta + \overline{\pi} - \overline{\alpha} + ie\mathbf{A} \cdot \mathbf{m})G_1 = i \frac{\mu_0}{\sqrt{2}}\, F_2$$

$$(\nabla + \overline{\mu} - \overline{\gamma} + ie\mathbf{A} \cdot \mathbf{n})G_1 - (\overline{\delta} + \overline{\beta} - \overline{\tau} + ie\mathbf{A} \cdot \overline{\mathbf{m}})G_2 = i \frac{\mu_0}{\sqrt{2}}\, F_1 \tag{30}$$

where F_1, F_2, G_1, and G_2 are the tetrad component spinors, D, ∇, δ, and $\overline{\delta}$ are the ordinary differentiation designations, and ε, ρ are spin coefficients defined by Newman and Penrose [10]

We take the signature $(+, -, -, -)$. From the metric (1) we have

$$n_\mu = (-B/2, 1, 0, 0)$$
$$m_\mu = -\frac{r}{\sqrt{2}}\,(0, 0, 1, i \sin\theta)$$
$$l_\mu = (-1, 0, 0, 0)$$
$$\overline{m}_\mu = -\frac{r}{\sqrt{2}}\,(0, 0, 1, -i \sin\theta)$$

We obtain the spin coefficients as in ref. 10:

$$\kappa = \lambda = \sigma = \nu = \tau = \pi = \varepsilon = 0$$
$$\rho = -1/r, \qquad \alpha = -\beta = -\frac{1}{2\sqrt{2}r}\,\mathrm{ctg}\,\theta \tag{31}$$
$$\mu = -B/2r, \qquad \gamma = B'/4 = 1/4\, \partial B/\partial r$$

We have

$$D = l^{\mu}\,\partial_{\mu} = \frac{\partial}{\partial r}, \qquad \nabla = n^{\mu}\,\partial_{\mu} = -\frac{\partial}{\partial v} - \frac{A\partial}{2\partial r}$$

$$\delta = m^{\mu}\partial_{\mu} = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{32}$$

$$\bar{\delta} = \bar{m}^{\mu}\,\partial_{\mu} = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

So Eqs. (29) and Eq. (30) reduce to

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)F_1 + \left[\frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + \frac{1}{2r\sqrt{2}}\operatorname{ctg}\theta\right]F_2 = \frac{i\mu_0}{\sqrt{2}}G_1 \tag{33}$$

$$\left[-\frac{\partial}{\partial v} - \frac{B\partial}{2\partial r} - \frac{1}{2r} + \frac{M}{2r^2} - \frac{ieQ}{r}\right]F_2 + \left[\frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + \frac{1}{2r\sqrt{2}}\operatorname{ctg}\theta\right]F_1 = \frac{i\mu_0}{\sqrt{2}}G_2 \tag{34}$$

$$\left[\frac{\partial}{\partial r} + \frac{1}{r}\right]G_2 - \left[\frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + \frac{1}{2r\sqrt{2}}\operatorname{ctg}\theta\right]G_1 = \frac{i\mu_0}{\sqrt{2}}F_2 \tag{35}$$

$$\left[-\frac{\partial}{\partial v} - \frac{B\partial}{2\partial r} - \frac{1}{2r} + \frac{M}{2r^2} - \frac{ieQ}{r}\right]G_1 - \left[\frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + \frac{1}{2r\sqrt{2}}\operatorname{ctg}\theta\right]G_2 = \frac{i\mu_0}{\sqrt{2}}F_1 \tag{36}$$

Separating variables as

$$F_1 = e^{in\varphi}\,rR_-(r, v)\,S_-(\theta), \qquad F_2 = e^{in\varphi}\,r\,R_+(r, v)S_+(\theta)$$
$$G_1 = e^{in\varphi}\,R_+(r, v)\,S_-(\theta), \qquad G_2 = e^{in\varphi}\,R_-(r, v)S_+(\theta) \tag{37}$$

we reduce Eqs. (33) and (34) to

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)rR_- - \lambda_1 R_+ = 0 \tag{38}$$

$$\left(\frac{\partial}{\partial\theta} + \frac{n}{\sin\theta} + \frac{1}{2}\operatorname{ctg}\theta\right)S_+ = -\lambda_{10}\,S_- \tag{39}$$

Equations (35) and (36) reduce to

$$\left(-\frac{\partial}{\partial v}-\frac{B}{2}\frac{\partial}{\partial r}-\frac{1}{2r}+\frac{M}{2r^2}-\frac{ieQ}{r}\right)rR_1-\lambda_2 R=0 \tag{40}$$

$$\left(\frac{\partial}{\partial\theta}-\frac{n}{\sin\theta}+\frac{1}{2}\,ctg\,\theta\right)S_-=-\lambda_{20}S_+ \tag{41}$$

where $\lambda_1=\lambda_{10}+i\mu_0/\sqrt{2}$, $\lambda_2=\lambda_{20}+i\mu_0/\sqrt{2}$, and λ_{10}, λ_{20} are respective separation constants.

From Eqs. (38), (40) and Eqs. (39), (41) we can obtain equations for $R_-(r, v)$, $R_+(r, v)$, $S_-(\theta)$, and $S_+(\theta)$, but what we seek is the radial part of Dirac equation.

From (38) and (40), we have

$$2\frac{\partial^2\rho_-}{\partial r\partial v}+B\frac{\partial^2\rho_-}{\partial r^2}+\left(C_1+\frac{2ieQ}{r}\right)\frac{\partial\rho_-}{\partial r}+\frac{2}{r}\frac{\partial\rho_-}{\partial v}+W_1\rho_-=0 \tag{42}$$

$$2\frac{\partial^2\rho_+}{\partial r\partial v}+B\frac{\partial^2\rho_+}{\partial r^2}+\left(C_2+\frac{2ieQ}{r}\right)\frac{\partial\rho_+}{\partial r}+\frac{4}{r}\frac{\partial\rho_+}{\partial v}+W_2\rho_+=0 \tag{43}$$

where

$$C_1=B/r+1/r-2M/r^2,\qquad C_2=C_1+B/r+B'+M/r^2$$

$$\rho_-=R_-/r,\qquad \rho_+=R_+/r,\qquad W_2=W_1+M/r^3$$

$$W_1=1/r-M/r^3+2\lambda/r^2+2ieQ/r^2$$

$$r_*=\ln[r-r_+(v)],\qquad v_*=\int[B'(r_+, v)+\omega_0/\omega-B_1\,i/(2\omega)]\,dv \tag{44}$$

where r_+ is the location of the event horizon. We have $\omega_0=eQ/r_+$, $B_1=[C_1-B'-B/r]_{r=r_+}$, and

$$B'(r_+, v)=\partial B/\partial r|_{r=r_+}=2M/r_+^2-2Q^2/r_+^3$$

Introducing the generalized tortoise coordinate transformation (44), we can rewrite (42) as

$$\begin{aligned}&\frac{B-2\dot r_+}{r-r_+}\frac{\partial^2\rho_-}{\partial r_*^2}+2\left[B'(r_+, v)+\frac{\omega_0}{\omega}-\frac{B_1}{2\omega}i\right]\frac{\partial^2\rho_-}{\partial r_*\partial v}\\&+\left[\frac{2\dot r_+-B}{r-r_+}-\frac{2\dot r_+}{r}+C_1+\frac{2ierQ}{r}\right]\frac{\partial\rho_-}{\partial r_*}\\&+\frac{2(r-r_+)}{r}\left[B'(r_+, v)+\frac{\omega_0}{\omega}-\frac{B_1}{2\omega}i\right]\frac{\partial\rho_-}{\partial v_*}+(r-r_+)W_1\,\rho_-=0\end{aligned} \tag{45}$$

Near the horizon, Eq. (45) can be reduced to

$$B'(r_+, v)\frac{\partial^2\rho_-}{\partial r_*^2} + 2\left[B'(r_+, v) + \frac{\omega_0}{\omega} - \frac{B_1}{2\omega}i\right]\frac{\partial^2\rho_-}{\partial r_*\partial v_*}$$

$$+ [B_1 + 2i\omega_0]\frac{\partial\rho_-}{\partial r_*} = 0 \tag{46}$$

The ingoing wave solution ρ_-^{in} and the outgoing wave solution ρ_-^{out} are

$$\rho_-^{in} = e^{-i\omega v_*} \tag{47}$$

$$\rho_-^{out} = e^{-i\omega v_*}e^{2i\omega r_*} \tag{48}$$

Making use of the above method, we can obtain the radiation spectrum and the radiation temperature with v_* as time coordinate

$$N_\omega = \frac{1}{e^{\omega/T_{v_*}} - 1} \tag{49}$$

$$T_{v_*} = \frac{1}{4\pi} \tag{50}$$

The outgoing wave solution, the radiation spectrum, and the radiation temperatre with v as time coordinate are

$$\rho_{out} = e^{-i\tilde{\omega}v}e^{2i(\tilde{\omega}-\omega')r_*/\kappa}e^{2\omega_1' r_*/\kappa} \tag{51}$$

$$\tilde{N_\omega} = \frac{1}{e^{(\tilde{\omega}-\omega_0^1)/T} + 1} \tag{52}$$

$$T = \frac{1}{4\pi v}\int B'(r_+, v)\,dv$$

$$= \frac{1}{2\pi v}\int\left[\frac{M(v)}{r_+^2} - \frac{Q^2(v)}{r_+^3}\right]dv \tag{53}$$

where

$$\kappa = \frac{1}{v}\int B'\,(r_+, v)\,dv, \qquad \omega_0 = \frac{1}{v}\int\omega_0\,dv, \qquad \omega_1' = \frac{1}{2v}\int B_1(r_+, v)\,dv$$

From (19), (23), (50), and (53), we see that, for a KG particle and a Dirac particle, the Hawking radiation temperature is different under two distinct time coordinates. With v_* as the time coordinate, the radiation temperature is a constant $1/4\pi$. With v as the time coordinate, the radiation temperature is

$$T_v = \frac{1}{2\pi v} \int \left(\frac{m(v)}{r_+^2} - \frac{Q^2(v)}{r_+^3} \right) dv \tag{54}$$

where r_+ is given by (11). Therefore, we can think of the Hawking radiation temperature as a compensation effect under the time-scale transformation [11]. When space–time returns to the steady state, (23) and (54) go back to the known result.

From the above, we see that, by finding directly the solutions of the KG equation and the Dirac equation, the equation which determines the location of the event horizon is automatically obtained, the Hawking radiation temperature is given, and the physical mechanism of producing Hawking radiation temperature is shown. In this calculation we avoid the difficulty of solving for the energy-momentum tensor and provide a brief way to study the quantum thermal effect of the nonstatic black hole.

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